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Characterization of orthogonal polynomials with respect to a functional

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Abstract

In this paper we study questions of existence, uniqueness and characterization of polynomials orthogonal with respect to a linear, not necessarily definite, functional $\mathscr{L}$ defined on the set of Laurent polynomials. First we characterize with the help of the function $F_{\mathscr{L}}(z) := \mathscr{L}((y+z)/(y-z))$ polynomials orthogonal with respect to $\mathscr{L}$. Using this characterization, which has wide applications, we are able to settle the question of existence and uniqueness of orthogonal polynomials. The uniquely determined orthogonal polynomials will be called *basic orthogonal polynomials.* It is to be pointed out that, in contrast to the real case, there are natural numbers n_μ such that there exist no polynomials of degree n_μ which are orthogonal with respect to $\mathscr{L}$, if $\mathscr{L}$ is indefinite and if we have "orthogonality-jumps" greater than 1. Furthermore the functional to which the basic orthogonal polynomials of the second kind are orthogonal is determined. Finally, we get explicit expressions for all basic orthogonal polynomials with respect to a "weight function" the support of which consists of several arcs of the unit circle, changes sign from arc to arc and has square root singularities at the boundary points of the arcs. These polynomials can be considered as the basic polynomials in describing and generating orthogonal polynomials with periodic reflection coefficients.

Keywords: Orthogonal polynomials; Polynomials of the second kind; Nondefinite functionals; Hermitian inner product; Basic integers

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1. Introduction and notation

If one is interested in characterizing and describing orthogonal polynomials with respect to a sign-changing weight function or more generally with respect to a not necessarily positive

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measure, it is of advantage to study these questions first for the more general case of polynomials orthogonal with respect to a linear functional and then to apply the results to the above special case, as it will be demonstrated in this paper.

Hence, in the following let $\{c_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers such that

$$c_j = \overline{c_{-j}}, \quad j \in \mathbb{N}_0 \quad \text{and} \quad \sum_{j=0}^{\infty} c_j z^j \text{ converges on } |z| \leqslant \varrho, \quad \varrho > 0. \tag{1.1}$$

Let us mention that most of the following results remain valid with some minor modifications even if we do not assume $c_j = \overline{c_{-j}}$. We restrict ourselves to the case (1.1) because statements become more transparent.

To the sequence in (1.1) we associate a linear functional $\mathscr{L}$ on the set of Laurent-series, which is defined for Laurent-polynomials by

$$\mathscr{L}\left(\sum_{j=k}^{l} b_j z^j\right) := \sum_{j=k}^{l} b_j c_{-j}, \quad k, l \in \mathbb{Z}, \ b_j \in \mathbb{C}. \tag{1.2}$$

Functionals of this kind, but mainly under the more restrictive assumption (1.7) below, have been first studied by Szegő, Akhiezer, Krein, Geronimus, Grenander, Freud [1, 8, 11, 14], and – what is interesting for our purposes – later on, mainly in connection with Padé approximation, for instance by Gragg, Brezinski, Bultheel, Van Barel, Gutknecht, Freund and Zha [4–6, 9, 13, 15].

Henceforth, let $\mathbb{P}_n^{\mathbb{C}}$ denote the set of complex polynomials of degree (abbreviated by ∂) less than or equal to n, $n \in \mathbb{N}_0$, and let $\mathbb{P}^{\mathbb{C}}$ be the set of all complex polynomials. In this paper we consider orthogonal polynomials with respect to linear functionals $\mathscr{L}$ of the above form. In this context a polynomial $P_n \in \mathbb{P}_n^{\mathbb{C}}$ of exact degree n is called orthogonal with respect to $\mathscr{L}$, if

$$\mathscr{L}(z^{-j} P_n) = 0 \quad \text{for } j = 0, \dots, n-1. \tag{1.3}$$

More general, let $l, m \in \mathbb{Z}$ and $p \in \mathbb{P}_n^{\mathbb{C}}$. If we write

$$\mathscr{L}(z^{-j} p) = 0 \quad \text{for } j \in [l, \dots, m] \text{ resp. } j \in [l, \dots, m), \text{ etc.}$$

then this means that p has an exact lower orthogonality order l, i.e., $\mathscr{L}(z^{-(l-1)}p) \neq 0$, and an exact upper orthogonality order m, i.e., $\mathscr{L}(z^{-(m+1)}p) \neq 0$, resp. exact lower orthogonality order l and upper orthogonality order m, but the exact upper orthogonality order may be higher. Thus square brackets indicate the exact order of orthogonality while round brackets indicate an orthogonality order, only.

Further, we say that a polynomial P_n of exact degree n has a *normal orthogonality property*, if

$$\mathscr{L}(z^{-j} P_n) = 0 \quad \text{for } j \in (0, \dots, n-1].$$

Obviously, a monic polynomial $P_n(z) = z^n + b_{n-1} z^{n-1} + \cdots + b_0$, $n \in \mathbb{N}_0$, fulfills an (at least normal) orthogonality property with respect to $\mathscr{L}$, if and only if there holds

$$\begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-n+1} \\ c_1 & c_0 & \cdots & c_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} c_{-n} \\ c_{-n+1} \\ \vdots \\ c_{-1} \end{pmatrix} \tag{1.4}$$

and it is uniquely determined by (1.3) resp. (1.4) if and only if

$$D_{n-1} := |T_{n-1}| \neq 0, \tag{1.5}$$

where T_{n-1} denotes the hermitian Toeplitz matrix in (1.4). If (1.5) is fulfilled then P_n is of the form

$$P_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} c_0 & c_{-1} & \cdots & c_{-n+1} & c_{-n} \\ c_1 & c_0 & \cdots & c_{-n+2} & c_{-n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & c_0 & c_{-1} \\ 1 & z & \cdots & z^{n-1} & z^n \end{vmatrix}, \quad n \in \mathbb{N}_0 \ (D_{-1} := 1). \tag{1.6}$$

As usual, we call a sequence $\{c_j\}_{j \in \mathbb{Z}}$ resp. a functional $\mathscr{L}$ of the form (1.2) *definite*, if

$$D_{n-1} \neq 0 \quad \text{for all } n \in \mathbb{N}. \tag{1.7}$$

Hence, if $\mathscr{L}$ is definite then there exists a uniquely determined sequence of monic orthogonal polynomials $\{P_n\}_{n \in \mathbb{N}_0}$, which are given by (1.6).

In this paper we mainly study nondefinite linear functionals $\mathscr{L}$ and we are mainly interested in the question of characterizing and describing the uniquely determined polynomials orthogonal with respect to $\mathscr{L}$. If one knows these uniquely determined polynomials, which we will call basic polynomials, one can give all the orthogonal polynomials with respect to $\mathscr{L}$ explicitly; cf. Proposition 2.4(c) below.

In order to get our results we need the following notations: to $\mathscr{L}$ we associate the function

$$F_{\mathscr{L}}(z) := \mathscr{L}\left(\frac{y+z}{y-z}\right), \quad \text{where } \mathscr{L} \text{ acts on } y. \tag{1.8}$$

By the second condition in (1.1) $F_{\mathscr{L}}$ is analytic at $z = 0$ and has the representation

$$F_{\mathscr{L}}(z) = c_0 + 2 \sum_{j=1}^{\infty} c_j z^j, \quad |z| \leqslant \varrho. \tag{1.9}$$

Since there is a one-to-one correspondence between $\mathscr{L}$ and $F_{\mathscr{L}}$, we often will use the notation

$$P_n \perp F_{\mathscr{L}} \quad \text{for } j \in (0, \ldots, n-1)$$

instead of (1.3).

For a given polynomial A of degree n with leading coefficient α the polynomial of the *second kind* of A with respect to $\mathscr{L}$ is defined by

$$B(z) := \begin{cases} \mathscr{L}\left(\dfrac{y+z}{y-z}(A(y) - A(z))\right), & n \in \mathbb{N}, \\ \alpha c_0, & n = 0, \end{cases} \quad \text{where } \mathscr{L} \text{ acts on } y. \tag{1.10}$$

Then $\partial B = \partial A - l$, where l is given by $c_0 = \cdots = c_{l-1} = 0$ and $c_l \neq 0$ ($\partial B < 0$ means that B vanishes identically).

As usual the *reciprocal* polynomial A^* of a polynomial A is defined by $A^*(z) := z^{\partial A}\bar{A}(1/z)$. In what follows we will also need the *modified reciprocal* polynomial $A_n^{(*)}$ of A with respect to $\mathbb{P}_n^{\mathbb{C}}$, $n \geqslant \partial A$, defined by

$$A_n^{(*)}(z) := z^n \bar{A}(1/z), \quad \text{i.e., } A_n^{(*)} = z^{n-\partial A} A^*. \tag{1.11}$$

Note that in the notation of (1.11) the lower index n has to appear explicitly. Furthermore, let us mention that $A^* = A_{\partial A}^{(*)}$. We call a polynomial A *selfreciprocal* (with respect to $\mathbb{P}_n^{\mathbb{C}}$) if there exists a constant $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that

$$A_n^{(*)} = \lambda A \quad \text{resp.} \quad A^* = \lambda A.$$

Finally, let $H(z) = \sum_{j=0}^{\infty} h_j z^j$ be an analytic function at $z = 0$, then we write

$$H(z) = O(z^\nu), \quad \text{if } h_0 = \cdots = h_{\nu-1} = 0 \quad \text{and} \quad H(z) = \dot{O}(z^\nu), \quad \text{if in addition } h_\nu \neq 0.$$

This paper is organized as follows: In Section 2 we prove a characterization theorem for polynomials orthogonal with respect to a linear functional. With the help of this result we are able to settle the question of existence and uniqueness of the orthogonal polynomials. The uniquely determined orthogonal polynomials will be called *basic (orthogonal) polynomials*. By the basic polynomials all the orthogonal polynomials with respect to $\mathscr{L}$ can be given in an explicit form. Furthermore, the functional to which the basic polynomials of the second kind are orthogonal is determined. Finally, in Section 3 we state an example of a nondefinite sequence of basic polynomials.

2. Characterization and properties of orthogonal polynomials with respect to $\mathscr{L}$

In this section we first give a necessary and sufficient condition for a polynomial to be orthogonal with respect to $\mathscr{L}$. This characterization theorem, as it already turned out [21, 23], is – among others – of importance in the description of polynomials orthogonal with respect to weight functions (distributions). In this paper we use the characterization theorem first of all to study the questions of existence and uniqueness of polynomials orthogonal with respect to a linear (nondefinite) functional $\mathscr{L}$ of the form (1.2), i.e., we consider, as well, polynomials satisfying a higher order of orthogonality in detail.

Theorem 2.1. *Let the linear functional $\mathscr{L}$ and the associated function $F_{\mathscr{L}}$ be given as in* (1.2) *and* (1.8), *respectively. Further let P_n be a complex polynomial of degree n, $n \in \mathbb{N}_0$, $\Omega_n \in \mathbb{P}_n^{\mathbb{C}}$ and let $\mu \in \mathbb{Z} \cup \{\infty\}$, $\mu > \min\{-n, -1\}$, and $\kappa \in \mathbb{N} \cup \{\infty\}$. Then there hold*

(a) *The following two statements are equivalent.*

(i) *P_n and Ω_n satisfy the conditions*

$$\begin{cases} P_n(z)F_{\mathscr{L}}(z) + \Omega_n(z) = \dot{O}(z^{n+\mu}) \\ P_n^*(z)F_{\mathscr{L}}(z) - \Omega_n^{(*)}(z) = \dot{O}(z^{n+\kappa}) \end{cases} \quad \text{as } z \to 0. \tag{2.1}$$

(ii) $$\mathscr{L}(z^{-j}P_n) = 0 \quad \text{for } j \in [-(\kappa-1), \ldots, n+\mu-1]$$

and Ω_n is the polynomial of the second kind of P_n with respect to $\mathscr{L}$.

(b) *Let* P_n *satisfy the higher orthogonality property* $\mathscr{L}(z^{-j}P_n) = 0$ *for* $j \in [-(\kappa - 1), \ldots, n + \mu - 1]$, *where* $\mu \in \mathbb{N}$. *Further, denote the multiplicity of the zero of* P_n *at* $z = 0$ *by* r, $r \in \mathbb{N}_0$. *Then there hold*

$$r \leqslant \mu - 1 \quad \textit{and} \quad \kappa = \mu - r.$$

Proof. (a) (i) $\Rightarrow$ (ii). We show the assertion only for $\mu \in \mathbb{N}_0$ and $\kappa \in \mathbb{N}$, because the other cases can be treated in the same way. Let us denote $P_n(z) = b_n z^n + \cdots + b_0$ and $\Omega_n(z) = d_n z^n + \cdots + d_0$. From both equations in (2.1) one gets by comparing coefficients, recall the $\dot{O}$-terms in (2.1), that

$$\sum_{k=0}^{n} b_k c_{j-k} = \mathscr{L}(z^{-j}P_n) = 0 \quad \text{for } j \in [-(\kappa - 1), \ldots, n + \mu - 1].$$

Hence, it remains to show that Ω_n is the polynomial of the second kind of P_n with respect to $\mathscr{L}$. Since $\kappa \geqslant 1$ the polynomial $\Omega_n^{(*)}$, and thus also Ω_n, is uniquely determined by $F_{\mathscr{L}}$, P_n^* and the second equation in (2.1). Therefore it suffices to show that the polynomial of the second kind of P_n with respect to $\mathscr{L}$, which is given by (1.10) and which we denote by $\tilde{\Omega}_n$, satisfies

$$P_n^*(z)F_{\mathscr{L}}(z) - \tilde{\Omega}_n^{(*)}(z) = O(z^{n+1}) \quad \text{as } z \to 0. \tag{2.2}$$

For $n = 0$ this is obvious. Let now $n \geqslant 1$. From the definition of $\tilde{\Omega}_n$ one gets

$$\tilde{\Omega}_n^{(*)}(z) = -\mathscr{L}\left(\frac{y+z}{y-z}(z^n\bar{P}_n(1/y) - P_n^*(z))\right)$$

and together with the linearity of $\mathscr{L}$ and (1.8) there follows

$$\begin{aligned}\tilde{\Omega}_n^{(*)}(z) &= -z^n\mathscr{L}\left(\left(1 + 2\sum_{k=1}^{\infty} z^k y^{-k}\right)\bar{P}_n\left(\frac{1}{y}\right)\right) + P_n^*(z)F_{\mathscr{L}}(z)\\ &= -z^n O(z) + P_n^*(z)F_{\mathscr{L}}(z) \quad \text{as } z \to 0,\end{aligned}$$

where the second equation holds by the already proven fact that $\mathscr{L}(\bar{P}_n(1/y)) = \overline{\mathscr{L}(P_n(y))} = 0$. This is property (2.2).

(ii) $\Rightarrow$ (i). Let P_n be an orthogonal polynomial with the given orthogonality property and let Ω_n be the corresponding polynomial of the second kind with respect to $\mathscr{L}$. Again the system (2.1) is obvious for $n = 0$. For $n \geqslant 1$ we get by the definition of Ω_n, by the linearity of $\mathscr{L}$ and by (1.8)

$$\begin{aligned}\Omega_n(z) &= \mathscr{L}\left(\left(1 + 2\sum_{k=1}^{\infty} z^k y^{-k}\right)P_n\right) - P_n(z)F_{\mathscr{L}}(z)\\ &= -P_n(z)F_{\mathscr{L}}(z) + \dot{O}(z^{n+\mu}) \quad \text{as } z \to 0,\end{aligned}$$

which is the first equation in (2.1). The second one follows in the same way as (2.2) by using the exact orthogonality property of P_n.

(b) By part (a) there holds a system of the form (2.1) and by the first equation in this system the polynomial Ω_n has a zero at $z = 0$ at least of multiplicity r. Let us denote $\Omega_n(z) =: z^r \Omega_{n-r}(z)$, $P_n(z) =: z^r P_{n-r}(z)$ where $\Omega_n^{(*)} = \Omega_{n-r}^{(*)}$, $P_n^* = P_{n-r}^*$, then there follows from (2.1)

$$\frac{\Omega_{n-r}(z)}{P_{n-r}(z)} + \frac{\Omega_{n-r}^{(*)}(z)}{P_{n-r}^*(z)} = O(z^{\min\{n-r+\mu, n+\kappa\}}) \quad \text{as } z \to 0. \tag{2.3}$$

We first consider the case $\mu \leqslant \kappa + r$. Then (2.3) yields

$$\Omega_{n-r}(z)P_{n-r}^*(z) + \Omega_{n-r}^{(*)}(z)P_{n-r}(z) = \begin{cases} d_{n-r+\mu}z^{n-r+\mu} + \cdots + d_{2n-2r}z^{2n-2r}, & \mu \leqslant n-r \\ 0, & \mu > n-r, \end{cases}$$

where $d_{n-r+\mu}, \ldots, d_{2n-2r} \in \mathbb{C}$. Taking into account that the polynomial on the left-hand side is *selfreciprocal* (with respect to $\mathbb{P}_{2n-2r}^{\mathbb{C}}$) there follows from the above representation

$$\Omega_{n-r}P_{n-r}^* + \Omega_{n-r}^{(*)}P_{n-r} \equiv 0. \tag{2.4}$$

This identity is, by (2.1) and (2.3), only possible if $n - r + \mu = n + \kappa$, i.e., $\mu = \kappa + r$, which is the assertion.

Applying the same arguments to the case $\mu > \kappa + r$, this also leads to the identity (2.4), but which now contradicts $\Omega_{n-r}P_{n-r}^* + \Omega_{n-r}^{(*)}P_{n-r} = \dot{O}(z^{n+\kappa})$ following from (2.3). Thus $\mu > \kappa + r$ is impossible and the corollary has been shown. □

As the referee informed us, Theorem 2.1 can be extracted from results in the field of Padé approximation. For instance, combining some results of Gutknecht [15, Section 7], the equivalence of the statements (i) and (ii) can be derived, but without the explicit description of the values μ and κ.

Let us give now some illustrative examples of (partially known) results, which can be shown in an elegant way by applying Theorem 2.1.

Example 2.2. (a) (Compare [12] for the positive definite case, i.e., $D_{n-1} > 0$ for all $n \in \mathbb{N}$.) If the function $F_{\mathscr{L}}$ has no zero at $z = 0$ (for instance if $F_{\mathscr{L}}$ generates a definite functional $\mathscr{L}$), then $\partial\Omega_n = n$ and Theorem 2.1(a) yields by multiplying by $1/F_{\mathscr{L}}$ that

$$\Omega_n \perp \frac{1}{F_{\mathscr{L}}} \quad \text{for } j \in [-(\kappa - 1), \ldots, n + \mu - 1].$$

The situation becomes more complicated if $F_{\mathscr{L}}(0) = 0$. We will treat this case later in Theorem 2.5.

(b) Let A be an arbitrary polynomial of degree $\partial A =: n_0$ which has no zero on the unit circle $|z| = 1$. Then there exists, up to a real factor, a uniquely determined polynomial B, $\partial B \leqslant n_0$, such that

$$A(z)B_{n_0}^{(*)}(z) + A^*(z)B(z) = \text{const.}\, z^{n_0}, \quad \text{const.} \in \mathbb{R}^+. \tag{2.5}$$

It follows immediately from Theorem 2.1(a) that the polynomials $P_n(z) := z^{n-n_0}A(z)$, $n \geqslant n_0$, satisfy the orthogonality property

$$P_n \perp F_{\mathscr{L}}(z) := \frac{B_{n_0}^{(*)}(z)}{A^*(z)} \quad \text{for } j \in (\ldots, -2, -1, 0, \ldots, n-1],$$

i.e., $\mu = 0$ and $\kappa = \infty$ in (2.1). If A has all its zeros in the open unit disk $|z| < 1$, then the P_n's, $n \geqslant n_0$, are the well-known Bernstein–Szegő polynomials (see for example [3, 24, p. 31]).

(c) (Compare Badkov [2] and later rediscovered by Marcellán and Sansigre [20] for $N = 2$ and then by Ismail and Li [17] for the general case.) Let $\{P_n\}_{n \in \mathbb{N}_0}$ be the sequence of monic orthogonal polynomials with respect to a definite linear functional $\mathscr{L}$ of the form (1.2). From Theorem 2.1(a), we immediately get that for each $N \in \mathbb{N}$ the polynomials

$$\tilde{P}_{mN+j}(z) := z^j P_m(z^N), \quad m \in \mathbb{N}_0,\ j \in \{0, \ldots, N-1\}$$

are orthogonal with respect to the functional $F_{\tilde{\mathscr{L}}}(z) := F_{\mathscr{L}}(z^N)$, i.e. with respect to the definite linear functional $\tilde{\mathscr{L}}$ given by

$$\tilde{\mathscr{L}}(z^{-(mN+j)}) := \begin{cases} \mathscr{L}(z^{-m}) = c_m, & m \in \mathbb{N}_0 \text{ and } j = 0, \\ 0, & m \in \mathbb{N}_0 \text{ and } j \in \{1, \ldots, N-1\}. \end{cases}$$

We now give, besides (1.5), some necessary and sufficient conditions for the uniqueness of orthogonal polynomials. Again, as pointed out by the referee, some of these statements could also be derived by results from the field of Padé approximation (cf. e.g. [6, 9, 13, 15]). Let us show how to get the results by applying Theorem 2.1.

Proposition 2.3. *Let $\mathscr{L}$ be given as in* (1.2), *let $n \in \mathbb{N}_0$ and let P_n be an orthogonal polynomial with respect to $\mathscr{L}$. The corresponding polynomial of the second kind is denoted by Ω_n.*

(a) *The following equivalence holds: P_n and Ω_n have no common zero in $\mathbb{C}\setminus\{0\}$ if and only if P_n is uniquely determined by* (1.3), *i.e. $D_{n-1} \neq 0$, or P_n is of the form $P_n(z) = z^r P_{n-r}(z)$, where P_{n-r} is a uniquely determined orthogonal polynomial with respect to $\mathscr{L}$.*

(b) *If P_n fulfills a normal orthogonality order, then P_n is uniquely determined.*

(c) *There exists an orthogonal polynomial P_n with a normal orthogonality order if and only if $D_{n-1} \cdot D_n \neq 0$.*

Proof. (a) *Sufficiency.* By Theorem 2.1(a) the polynomials P_n and Ω_n fulfill the conditions

$$\begin{cases} P_n(z)F_{\mathscr{L}}(z) + \Omega_n(z) = O(z^n) \\ P_n^*(z)F_{\mathscr{L}}(z) - \Omega_n^{(*)}(z) = O(z^{n+1}) \end{cases} \quad \text{as } z \to 0. \tag{2.6}$$

Assume that P_n is uniquely determined by (1.3). If P_n and Ω_n had a common zero at $z_0 \neq 0$ then every pair of polynomials of the form $A = ((z - z_1)/(z - z_0))\, P_n$, $B = ((z - z_1)/(z - z_0))\, \Omega_n$, where $z_1 \in \mathbb{C}$ is arbitrary, would solve (2.6). Thus, again by Theorem 2.1(a), for every $z_1 \neq z_0$ the polynomial A, $A \neq P_n$, would be another orthogonal polynomial of degree n with respect to $\mathscr{L}$.

If P_n is of the form $P_n(z) = z^r P_{n-r}(z)$, where P_{n-r} is a uniquely determined orthogonal polynomial, then it can be seen from (2.1) that the polynomial of the second kind is also of the form $\Omega_n(z) =: z^r \Omega_{n-r}(z)$, where Ω_{n-r} denotes the polynomial of the second kind of P_{n-r}. Now again the assertion follows as above.

Necessity. Let $r \in \mathbb{N}_0$ denote the multiplicity of the zero of P_n at $z = 0$, i.e., $P_n(z) =: z^r P_{n-r}(z)$, where $P_{n-r} \in \mathbb{P}_{n-r}^{\mathbb{C}}$ and $P_{n-r}(0) \neq 0$. As above we see that Ω_n is also of the form $\Omega_n(z) =: z^r \Omega_{n-r}(z)$

and $\Omega_{n-r} \in \mathbb{P}_{n-r}^{\mathbb{C}}$. Now from (2.6) we obtain

$$-\frac{\Omega_{n-r}(z)}{P_{n-r}(z)} = F_{\mathscr{L}}(z) + O(z^{n-r}) \quad \text{and} \quad \frac{\Omega_{n-r}^{*}(z)}{P_{n-r}^{*}(z)} = F_{\mathscr{L}}(z) + O(z^{n+1}). \tag{2.7}$$

Now suppose that A_n with $A_n(z) =: z^s A_{n-s}(z)$, $A_{n-s}(0) \neq 0$, $s \in \mathbb{N}_0$, is another polynomial of degree n with the same leading coefficient as P_n, which fulfills (1.3). Let B_n be the corresponding polynomial of the second kind. Then there follows (note that again B_n can be written as $B_n(z) =: z^s B_{n-s}(z)$)

$$-\frac{B_{n-s}(z)}{A_{n-s}(z)} = F_{\mathscr{L}}(z) + O(z^{n-s}) \quad \text{and} \quad \frac{B_{n-s}^{(*)}(z)}{A_{n-s}^{*}(z)} = F_{\mathscr{L}}(z) + O(z^{n+1}). \tag{2.8}$$

As a consequence of (2.7) and (2.8) there can be derived, similar as in (2.4), that

$$\Omega_{n-r} A_{n-s} \equiv B_{n-s} P_{n-r}.$$

By assumption P_{n-r} and Ω_{n-r} have no zero in common. If A_{n-s} and B_{n-s} have also no zero in common, then from the above identity there follows $A_{n-s} = P_{n-r}$ (note that A_n and P_n have the same leading coefficient) and $B_{n-s} = \Omega_{n-r}$, which is the assertion. If, in the other case, A_{n-s} and B_{n-s} have common zeros, i.e., $n - s > n - r$ and thus $s < r$ by the above identity, then by applying the same procedure on P_{n-r} instead of on P_n one gets the uniqueness of P_{n-r}.

(b) The system (2.1) in Theorem 2.1(a) takes now the form (2.6), where in the first equation there even holds $\dot{O}(z^n)$. From this system one can obtain by similar methods as in the proof of Theorem 2.1(b) that

$$\Omega_n(z)P_n^*(z) + \Omega_n^{(*)}(z)P_n(z) = \text{const. } z^n, \quad \text{const.} \neq 0.$$

Thus the polynomials P_n and Ω_n cannot vanish simultaneously in $\mathbb{C}\setminus\{0\}$ and the assertion follows from part (a).

(c) *Sufficiency.* Let $P_n(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_0$ be given as in (1.6), note that $D_{n-1} \neq 0$, and suppose $\mathscr{L}(z^{-n}P_n) = 0$ then $b_0 c_n + b_1 c_{n-1} + \cdots + c_0 = 0$, which yields together with (1.4) that the $n + 1$ columns of T_n are linearly dependent, i.e. $D_n = 0$, and which contradicts the assumption.

Necessity. By part (b) we have $D_{n-1} \neq 0$. Assume that $D_n = 0$, then we have rk $T_n = n$, where rk denotes the rank of a matrix. Thus the null-space of T_n is of dimension 1 and by (1.4) and $D_{n-1} \neq 0$ the vector $(b_0, \ldots, b_{n-1}, 1)^{\mathrm{T}}$ is in this null-space. This implies that $\mathscr{L}(z^{-n}P_n) = 0$, which is the desired contradiction. □

In what follows we will call an orthogonal polynomial P_n, which is uniquely determined, i.e., $D_{n-1} \neq 0$, a *basic* (*orthogonal*) *polynomial* with respect to $\mathscr{L}$. For instance, by Proposition 2.3(b) all polynomials satisfying a normal orthogonality property are basic polynomials. Further, Proposition 2.3(c) shows that if P_n fulfills a normal orthogonality property then there even exist the two successive uniquely determined monic orthogonal polynomials P_n and P_{n+1}, since $D_{n-1} \neq 0$ and $D_n \neq 0$.

We now consider the more complicated case $D_{n-1} \neq 0$ and $D_n = 0$, which can appear as it was shown e.g. in [1, Section 2] or [16, Remark 1, p. 98], in detail. This means that there exists the basic polynomial P_n with respect to $\mathscr{L}$, which fulfills a higher order of orthogonality by Proposition 2.3(c).

Parts of the following proposition could be obtained directly by results of [7, 13, 15]. To make the paper selfcontained, let us give a compact independent proof.

Proposition 2.4. *Let the complex sequence $\{c_j\}_{j\in\mathbb{Z}}$ resp. the linear functional $\mathscr{L}$ be given as in* (1.1) *and* (1.2), *respectively. Assume that $D_{n-1}\neq 0$ and $D_n = 0$ and let $n+\mu-1$, $\mu\in\mathbb{N}\cup\{\infty\}$, be the exact upper order of orthogonality of the polynomial P_n from* (1.6). *As usual, Ω_n denotes the polynomial of the second kind of P_n with respect to $\mathscr{L}$. Then there hold:*

(a) *$P_n(0)\neq 0$ and $P_n^* = cP_n$, $\Omega_n^{(*)} = -c\Omega_n$, where $c := \overline{P_n(0)}$ with $|c| = 1$.*

(b) *The lowest degree greater than n for which there exists a uniquely determined orthogonal polynomial with respect to $\mathscr{L}$ is $n+2\mu$.*

(c) *Let $1\leqslant k\leqslant \mu-1$ and let P_{n+k} be a polynomial of exact degree $n+k$. Then P_{n+k} is an orthogonal polynomial with respect to $\mathscr{L}$ if and only if P_{n+k} is of the form*

$$P_{n+k}(z) = z^r g_{k-r}(z)P_n(z), \tag{2.9}$$

where $g_{k-r}\in\mathbb{P}_{k-r}^{\mathbb{C}}$ and $g_{k-r}(0)\neq 0$. In this case, P_{n+k} satisfies

$$\mathscr{L}(z^{-j}P_{n+k}) = 0 \quad \text{for } j\in[-(\mu-k-1),\dots,n+\mu+r-1]. \tag{2.10}$$

(d) *There exists no polynomial $p\in\mathbb{P}_{n+2\mu-1}^{\mathbb{C}}\backslash\mathbb{P}_{n+\mu-1}^{\mathbb{C}}$ which is orthogonal with respect to $\mathscr{L}$.*

Proof. (a) By (1.6) we have

$$P_n(0) = \frac{(-1)^n}{D_{n-1}}\begin{vmatrix} c_{-1} & \cdots & c_{-n} \\ \vdots & \ddots & \vdots \\ c_{n-2} & \cdots & c_{-1}\end{vmatrix},$$

which is unequal to zero by $D_{n-1}\neq 0$, $D_n = 0$ and [16, Proposition 1°, p. 103]). By assumption, by $P_n(0)\neq 0$ and by Theorem 2.1(b) we have $\mathscr{L}(z^{-j}P_n) = 0$ for $\mu\in[-(\mu-1),\dots,n+\mu-1]$, i.e., by Theorem 2.1(a)

$$\begin{cases} P_n(z)F_{\mathscr{L}}(z) + \Omega_n(z) = \dot{O}(z^{n+\mu}) \\ P_n^*(z)F_{\mathscr{L}}(z) - \Omega_n^{(*)}(z) = \dot{O}(z^{n+\mu}) \end{cases} \quad \text{as } z\to 0.$$

For each $c\in\mathbb{C}\backslash\{0\}$ the pair $((1/c)P_n^*, -(1/c)\Omega_n^{(*)})$ satisfies this system, too, such that by the uniqueness of P_n we get $P_n^* = cP_n$ and $\Omega_n^{(*)} = -c\Omega_n$. With the notation $b_0 := P_n(0)\neq 0$ there further follows that $c = \bar{b}_0$ and $cb_0 = 1$, thus $|c| = 1$.

(b) From the orthogonality property of P_n there follows that $D_{n-1}\neq 0$ and $D_n = \cdots = D_{n+2(\mu-1)} = 0$. By using the so-called (r,k)-characteristic, introduced by Iohvidov [16], one can derive that $D_{n+2\mu-1}\neq 0$ (compare especially [16, Theorems 15.1 and 15.6]). This gives by (1.6) the assertion.

(c) *Sufficiency.* Let P_{n+k} be of the form (2.9). Then (2.10), and thus the fact that P_{n+k} is an orthogonal polynomial with respect to $\mathscr{L}$, follows from

$$\begin{cases} z^r g_{k-r}(z)P_n(z)F_{\mathscr{L}}(z) + z^r g_{k-r}(z)\Omega_n(z) = \dot{O}(z^{n+k+(\mu-k+r)}) \\ g_{k-r}^*(z)P_n^*(z)F_{\mathscr{L}}(z) - g_{k-r}^*(z)\Omega_n^{(*)}(z) = \dot{O}(z^{n+k+(\mu-k)}) \end{cases} \quad \text{as } z\to 0$$

and Theorem 2.1(a), note that $\mu-k\geqslant 1$.

Necessity. Because the methods are so similar we will give a joint proof of (d) and of the necessary part of (c). This means, we suppose that P_m is an orthogonal polynomial with respect to $\mathscr{L}$ of exact degree m where $m \in \{n+1, \ldots, n+2\mu-1\}$. We first show that only $n+1 \leqslant m \leqslant n+\mu-1$ is possible, this is part (d), and then we derive the representation (2.9) of P_{n+k}, where we write $m = n+k$, $1 \leqslant k \leqslant \mu - 1$.

Let r denote the multiplicity of the zero of P_m at $z=0$. Because of part (b) P_m is not uniquely determined by (1.3), thus it fulfills a higher order of orthogonality by Proposition 2.3(b), say (compare Theorem 2.1(b))

$$\mathscr{L}(z^{-j}P_m) = 0 \quad \text{for } j \in [-(\eta - r - 1), \ldots, m + \eta - 1] \text{ and } \eta \geqslant r+1. \tag{2.11}$$

Let Ω_m be the polynomial of the second kind of P_m with respect to $\mathscr{L}$, then by Proposition 2.3(a) the polynomial g_k, defined by $g_k := \gcd(P_m, \Omega_m)$, has a degree $k \geqslant 1$ and a zero at $z=0$ of exact multiplicity r (by the first equation in (2.1)). Let us define $g_{k-r} := g_k/z^r$ and

$$\tilde{P}_{m-k} := \frac{P_m}{z^r g_{k-r}} \quad \text{and} \quad \tilde{\Omega}_{m-k} := \frac{\Omega_m}{z^r g_{k-r}}. \tag{2.12}$$

One easily shows with the help of Theorem 2.1(a) that $\tilde{\Omega}_{m-k}$ is the polynomial of the second kind of $\tilde{P}_{m-k}$ with respect to $\mathscr{L}$ and that $\tilde{P}_{m-k}$ fulfills the orthogonality property

$$\mathscr{L}(z^{-j}\tilde{P}_{m-k}) = 0 \quad \text{for } j \in [-(\eta + k - r - 1), \ldots, (m-k) + (\eta + k - r) - 1]. \tag{2.13}$$

Since $\tilde{P}_{m-k}$ and $\tilde{\Omega}_{m-k}$ have no zero in common it follows from Proposition 2.3(a) that $\tilde{P}_{m-k}$ is a uniquely determined orthogonal polynomial with respect to $\mathscr{L}$. Thus with the help of (b) one can show that $\tilde{P}_{m-k} = P_n$, i.e., $n = m-k$ and $\mu = \eta + k - r$ by (2.13). This means, note (2.11), that $m = n+k = n+\mu-\eta+r \leqslant n+\mu-1$ and part (d) follows. Writing $m = n+k$ then (2.12) completes the proof of part (c). $\square$

Let us mention that from (2.10) follows again the, already in Proposition 2.3(b) proven, fact that the orthogonal polynomials P_{n+k}, $1 \leqslant k \leqslant \mu - 1$, with respect to $\mathscr{L}$, which are not uniquely determined by Proposition 2.4(b), fulfill a higher order of orthogonality. Further, it is remarkable that the upper orthogonality order in (2.10) does not depend on the degree of the orthogonal polynomial but only on the multiplicity r of the zero at $z=0$.

By Proposition 2.3(c) and Proposition 2.4(b) we get that for every linear functional $\mathscr{L}$ of the form (1.2) there exists a uniquely determined sequence of integers $\{n_\nu\}_{\nu\in\mathbb{N}_0}$, the so-called basic integers, such that for each basic integer there exists a monic basic polynomial $P_{n_\nu}(z) = z^{n_\nu} + \cdots$. To be more precise, this sequence of basic integers is either finite, i.e., there exists an index $N \in \mathbb{N}$ such that $n_N = n_{N+1} = \cdots$ (this is the case when there exists an orthogonal polynomial satisfying an infinite upper orthogonality property), or is infinite and satisfies $0 = n_0 < n_1 < \cdots$. Further, for the basic polynomials there holds

$$\mathscr{L}(z^{-j}P_{n_\nu}) = 0 \quad \text{for } j \in \begin{cases} [-(\mu_\nu - 1), \ldots, n_\nu + \mu_\nu - 1], & \text{for } \mu_\nu \geqslant 1, \\ (0, \ldots, n_\nu - 1], & \text{for } \mu_\nu = 0, \end{cases} \tag{2.14}$$

where $\mu_\nu := (n_{\nu+1} - n_\nu)/2 \in \mathbb{N}$ if $n_{\nu+1} - n_\nu \geqslant 2$ and $\mu_\nu := 0$ if $n_{\nu+1} = n_\nu + 1$.

As we have seen in Example 2.2(a) the polynomial of the second kind Ω_{n_ν} of a basic polynomial P_{n_ν}, $\nu \in \mathbb{N}_0$, is orthogonal with respect to the function $F_{\tilde{\mathscr{L}}} := 1/F_{\mathscr{L}}$ if $F_{\mathscr{L}}(0) \neq 0$. In the case of

$F_{\mathscr{L}} = \dot{O}(z^{\mu_0})$, $\mu_0 \in \mathbb{N}$, the situation is more complicated. Note that now the first basic polynomial $P_0(z) = 1$ fulfills $\mathscr{L}(z^{-j}P_0) = 0$ for $j \in [-\mu_0 + 1, \ldots, \mu_0 - 1]$ and Ω_{n_ν} is of degree $n_\nu - \mu_0$ by the considerations after (1.10).

Thus, let $\mu_0 \in \mathbb{N}$ and let us consider the following power series at $z = 0$

$$F_{\mathscr{L}}(z) =: z^{\mu_0}\hat{F}_{\mathscr{L}}(z) = \sum_{j=\mu_0}^{\infty} c_j z^j, \quad c_{\mu_0} \neq 0, \qquad \text{and} \qquad \frac{1}{\hat{F}_{\mathscr{L}}(z)} =: \sum_{j=0}^{\infty} d_j z^j, \quad d_j \in \mathbb{C}. \tag{2.15}$$

Let us further define the polynomial

$$\Phi(z) := \sum_{j=0}^{\mu_0 - 1} (d_j z^j - \bar{d}_j z^{2\mu_0 - j}) + \mathrm{i}\,\mathrm{Im}\{d_{\mu_0}\} z^{\mu_0} \in \mathbb{P}_{2\mu_0}^{\mathbb{C}}. \tag{2.16}$$

One easily sees

$$\partial\Phi = 2\mu_0, \qquad \Phi^* = -\Phi, \qquad \frac{1}{\hat{F}_{\mathscr{L}}} - \Phi = O(z^{\mu_0}), \qquad \left.\frac{(1/\hat{F}_{\mathscr{L}}) - \Phi}{z^{\mu_0}}\right|_{z=0} \in \mathbb{R}. \tag{2.17}$$

We now can state the following result.

Theorem 2.5. *Let $\{n_\nu\}_{\nu \in \mathbb{N}_0}$ be the sequence of basic integers with respect to a linear functional $\mathscr{L}$ and suppose that $\mu_0 \in \mathbb{N}$. The function $\hat{F}_{\mathscr{L}}$ and the polynomial Φ are given as in* (2.15) *and* (2.16), *respectively. As usual Ω_{n_ν}, $\nu \in \mathbb{N}_0$, denotes the polynomial of the second kind of the basic polynomial P_{n_ν} with respect to $\mathscr{L}$. Then*

$$\tilde{\Omega}_{n_\nu - 2\mu_0}(z) := \frac{1}{2\overline{c_{\mu_0}} z^{\mu_0}} \Omega_{n_\nu}(z) = z^{n_\nu - 2\mu_0} + \cdots, \quad \nu \in \mathbb{N}, \tag{2.18}$$

are the basic polynomials with respect to the function

$$F_{\tilde{\mathscr{L}}}(z) := \frac{(1/\hat{F}_{\mathscr{L}}) - \Phi(z)}{z^{\mu_0}} \tag{2.19}$$

and fulfill

$$\tilde{\Omega}_{n_\nu - 2\mu_0} \perp F_{\tilde{\mathscr{L}}} \quad \textit{for } j \in [-(\mu_\nu - 1), \ldots, (n_\nu - 2\mu_0) + \mu_\nu - 1],$$

where the μ_ν's are given as in (2.14).

Proof. We have $n_1 = 2\mu_0$ and $n_\nu > 2\mu_0$, $\nu \geqslant 2$, and by (1.10) one can obtain that $\Omega_{n_\nu}(z) = 2\overline{c_{\mu_0}} z^{n_\nu - \mu_0} + \cdots$ (note that P_{n_ν} is monic). Further, there follows from Proposition 2.4(a) that Ω_{n_ν} has a zero at $z = 0$ of multiplicity μ_0. Hence, $\tilde{\Omega}_{n_\nu - 2\mu_0}$, given in (2.18), is a monic polynomial of degree $n_\nu - 2\mu_0 \in \mathbb{N}_0$. From (2.1) we get

$$\frac{1}{2\overline{c_{\mu_0}}} P_{n_\nu}(z) + \frac{1}{\hat{F}_{\mathscr{L}}(z)} \tilde{\Omega}_{n_\nu - 2\mu_0}(z) = \dot{O}(z^{n_\nu + \mu_\nu - \mu_0}) \quad \text{as } z \to 0. \tag{2.20}$$

If we write

$$\frac{1}{2\overline{c_{\mu_0}}} P_{n_\nu}(z) + \Phi(z)\tilde{\Omega}_{n_\nu - 2\mu_0}(z) = \left(\frac{1}{2\overline{c_{\mu_0}}} P_{n_\nu}(z) + \frac{1}{\hat{F}_{\mathscr{L}}(z)} \tilde{\Omega}_{n_\nu - 2\mu_0}(z)\right) - \tilde{\Omega}_{n_\nu - 2\mu_0}(z)\left(\frac{1}{\hat{F}_{\mathscr{L}}(z)} - \Phi(z)\right),$$

we obtain by (2.17) and (2.20) that the polynomial at the left-hand side has a zero at least of order μ_0 at $z = 0$. Further, from

$$\left[\frac{1}{2\overline{c_{\mu_0}}} P_{n_\nu}(z) + \Phi(z)\tilde{\Omega}_{n_\nu - 2\mu_0}(z)\right]_{n_\nu}^{(*)} = \frac{1}{2c_{\mu_0}}\left[P_{n_\nu}^*(z) - \Phi(z)\frac{1}{z^{\mu_0}}\Omega_{n_\nu}^{(*)}(z)\right]$$

$$= \frac{1}{2c_{\mu_0}\hat{F}_{\mathscr{L}}(z)}\cdot\frac{1}{z^{\mu_0}}[P_{n_\nu}^*(z)F_{\mathscr{L}}(z) - \Omega_{n_\nu}^{(*)}(z)] + O(z^{\mu_0}) \quad \text{(by (2.17))}$$

$$= O(z^{\mu_0}) \quad \text{(by (2.1))},$$

there follows that $\partial[(1/(2\overline{c_{\mu_0}}))P_{n_\nu} + \Phi\tilde{\Omega}_{n_\nu - 2\mu_0}] \leqslant n_\nu - \mu_0$. By the above considerations we can define the polynomial

$$\tilde{P}_{n_\nu - 2\mu_0} := \frac{1}{z^{\mu_0}}\left(\frac{1}{2\overline{c_{\mu_0}}} P_{n_\nu} + \Phi\tilde{\Omega}_{n_\nu - 2\mu_0}\right) \in \mathbb{P}_{n_\nu - 2\mu_0}^{\mathbb{C}} \tag{2.21}$$

and from (2.18) and (2.21) one gets

$$\tilde{\Omega}_{n_\nu - 2\mu_0}(z)F_{\tilde{\mathscr{L}}}(z) + \tilde{P}_{n_\nu - 2\mu_0}(z) = \dot{O}(z^{(n_\nu - 2\mu_0) + \mu_\nu})$$

$$\tilde{\Omega}_{n_\nu - 2\mu_0}^*(z)F_{\tilde{\mathscr{L}}}(z) - \tilde{P}_{n_\nu - 2\mu_0}^{(*)}(z) = \begin{Bmatrix} \dot{O}(z^{(n_\nu - 2\mu_0) + \mu_\nu}), & \mu_\nu \geqslant 1 \\ O(z^{(n_\nu - 2\mu_0) + 1}), & \mu_\nu = 0 \end{Bmatrix} \quad \text{as } z \to 0.$$

Now the assertion follows. □

Remark. Note that the assertions of Theorem 2.5 even remain valid if Ω_n, $n \geqslant 2\mu_0$, is the polynomial of the second kind of an orthogonal polynomial P_n with respect to $\mathscr{L}$ which is not a basic polynomial.

3. Example: Basic polynomials on several arcs of the unit circle

We now give an example of a special sequence of basic polynomials orthogonal with respect to a sign-changing weight function, the support of which consists of several intervals and which has square root singularities at the end points of the intervals. These polynomials can be considered as the basic polynomials in describing and generating orthogonal polynomials with periodic reflection coefficients (compare [22]).

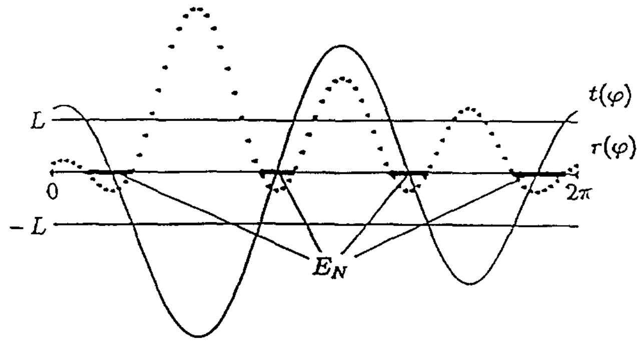

Fig. 1.

Let, therefore,

$$t(\varphi) = \sum_{j=0}^{N/2} r_j \cos \frac{N-2j}{2}\varphi + s_j \sin \frac{N-2j}{2}\varphi, \quad N \in \mathbb{N} \text{ and } N \text{ even},$$

be an arbitrary real trigonometric polynomial of degree $\frac{1}{2}N$, which has N simple zeros in $[0, 2\pi[$. Further let $L \in \mathbb{R}^+$ with $L < \min\{|t(\varphi)| : t'(\varphi) = 0\}$ and define

$$E_N := \{\varphi \in [a, a + 2\pi] : |t(\varphi)| \leqslant L\}, \tag{3.1}$$

where the real constant a is chosen such that $|t(a)| > L$. Then the set E_N consists of N disjoint intervals and we can write

$$E_N =: \bigcup_{j=1}^{N} [\varphi_{2j-1}, \varphi_{2j}]. \tag{3.2}$$

Finally, let $r(\varphi)$ denote that (up to a constant factor) uniquely determined real trigonometric polynomial of degree N, which vanishes at the boundary points φ_j of E_N and satisfies $r(\varphi) \leqslant 0$ for $\varphi \in E_N$ (compare Fig. 1, where $N = 4$ and $a = 0$).

In order to get uniqueness of r we choose the normalization factor such that

$$r(a) = t^2(a) - L^2 > 0 \quad (\text{note } a \notin E_N). \tag{3.3}$$

By this normalization, the trigonometric polynomials $t(\varphi)$ and $r(\varphi)$ are related by

$$t^2(\varphi) - r(\varphi) = L^2. \tag{3.4}$$

This identity follows immediately from the fact that the trigonometric polynomial on the left-hand side is of degree less than or equal to N and takes the value L^2 at the $(2N + 1)$ pairwise distinct points $a, \varphi_1, \ldots, \varphi_{2N} \in [a, a + 2\pi)$.

Now we define the selfreciprocal complex algebraic polynomials $\mathcal{T}_N$ and R_{2N} by

$$\mathcal{T}_N(e^{i\varphi}) := e^{i(N/2)\varphi} t(\varphi), \qquad R_{2N}(e^{i\varphi}) := e^{iN\varphi} r(\varphi). \tag{3.5}$$

From (3.4) and (3.5) we obtain

$$\mathcal{T}_N^2(z) - R_{2N}(z) = L^2 z^N \quad \text{for all } z \in \mathbb{C}, \tag{3.6}$$

and as a special consequence with $\alpha := \mathcal{T}_N(0)$

$$R_{2N}(0) = \mathcal{T}_N^2(0) = \alpha^2 \neq 0. \tag{3.7}$$

The following lemma shows that there exists an infinite sequence of polynomials satisfying an equation of the form (3.6).

Lemma 3.1. *Let T_m, U_{m-1}, $m \in \mathbb{N}$, be the classical Chebyshev polynomials of first resp. second kind, i.e., $T_m(x) = \cos(m \arccos \varphi)$, $U_{m-1}(x) = \sin(m \arccos \varphi)/\sin(\arccos \varphi)$, $x = \cos \varphi$, and let*

$$\mathcal{T}_{mN}(z) := \frac{1}{2^{m-1}} (Lz^{N/2})^m T_m\left(\frac{\mathcal{T}_N(z)}{Lz^{N/2}}\right),$$

$$\mathcal{U}_{(m-1)N}(z) := \pm \frac{1}{2^{m-1}} (Lz^{N/2})^{m-1} U_{m-1}\left(\frac{\mathcal{T}_N(z)}{Lz^{N/2}}\right).$$

Then $\mathcal{T}_{mN}(z) = \alpha^m z^{mN} + \cdots$ and $\mathcal{U}_{(m-1)N}(z) = \pm \alpha^{m-1} z^{(m-1)N} + \cdots$ are selfreciprocal polynomials which are related by

$$\mathcal{T}_{mN}^2(z) - R_{2N}(z)\mathcal{U}_{(m-1)N}^2(z) = \left(\frac{L^m}{2^{m-1}}\right)^2 z^{mN}. \tag{3.8}$$

Proof. Since $T_m(x) = 2^{m-1}x^m + \cdots$ (and $U_{m-1}(x) = 2^{m-1}x^{m-1} + \cdots$) is a real even (odd) polynomial if m is even resp. an odd (even) polynomial if m is odd it can be seen that $\mathcal{T}_{mN}$ and $\mathcal{U}_{(m-1)N}$ are selfreciprocal polynomials with leading coefficients α^m and $\pm\alpha^{m-1}$, respectively. Thus only the quadratic equation (3.8) remains to be shown.

For $\varphi \in E_N$ we have

$$\frac{\mathcal{T}_N(e^{i\varphi})}{Le^{i(N/2)\varphi}} = \frac{t(\varphi)}{L} \quad \text{and} \quad \left|\frac{t(\varphi)}{L}\right| \leqslant 1.$$

Thus

$$\begin{aligned} &\mathcal{T}_{mN}^2(e^{i\varphi}) - R_{2N}(e^{i\varphi})\mathcal{U}_{(m-1)N}^2(e^{i\varphi}) \\ &= \frac{1}{4^{m-1}} (L^2 e^{iN\varphi})^{m-1} \left[L^2 e^{iN\varphi} T_m^2\left(\frac{t(\varphi)}{L}\right) - R_{2N}(e^{i\varphi}) U_{m-1}^2\left(\frac{t(\varphi)}{L}\right)\right] \\ &= \frac{1}{4^{m-1}} (L^2 e^{iN\varphi})^{m} \left[T_m^2\left(\frac{t(\varphi)}{L}\right) - \left(\frac{t^2(\varphi)}{L^2} - 1\right) U_{m-1}^2\left(\frac{t(\varphi)}{L}\right)\right] \quad \text{(by (3.6))} \\ &= \frac{1}{4^{m-1}} (L^2 e^{iN\varphi})^{m}, \end{aligned}$$

where for the last equation we have used the well-known identity $T_m^2(x) - (x^2 - 1)U_{m-1}^2(x) = 1$ for $|x| \leqslant 1$. Since E_N contains infinitely many points, the identity (3.8) follows. □

We now choose the sign of $\mathscr{U}_{(m-1)N}$ such that there holds by (3.8)

$$\mathscr{T}_{mN}(z)\frac{\mathrm{i}z^{N/2}}{\sqrt{R_{2N}(z)}}+\mathrm{i}z^{N/2}\mathscr{U}_{(m-1)N}(z)=\dot{O}(z^{mN+(N/2)})\quad\text{as } z\to 0, \tag{3.9}$$

where we choose that branch of the square root $\sqrt{R_{2N}}$, which is analytic on $\mathbb{C}\setminus\{\mathrm{e}^{\mathrm{i}\varphi}:\varphi\in E_N\}$ and satisfies (compare [21, (2.1)])

$$\operatorname{sgn}\sqrt{R_{2N}(\mathrm{e}^{\mathrm{i}\varphi})}=(-1)^j\mathrm{e}^{\mathrm{i}(N/2)\varphi}\quad\text{for } \varphi\in(\varphi_{2j},\varphi_{2j+1}),\ j=0,\dots,N\quad(\varphi_0:=a,\ \varphi_{2N+1}:=a+2\pi).$$

Then we can state the following result.

Theorem 3.2. *Let the polynomials $\mathscr{T}_{mN}$, $\mathscr{U}_{(m-1)N}$, $m\in\mathbb{N}$, ($\mathscr{T}_0:=1$) be given as in Lemma 3.1 and define*

$$\frac{1}{f(\varphi)}:=\frac{(-1)^j}{\sqrt{|r(\varphi)|}}\quad\textit{for } \varphi\in[\varphi_{2j-1},\varphi_{2j}],\ j=1,\dots,N.$$

Then there hold for all $m\in\mathbb{N}_0$ in (3.10) resp. for all $m\in\mathbb{N}$ in (3.11)

$$\int_{E_N}\mathrm{e}^{-\mathrm{i}j\varphi}\mathscr{T}_{mN}(\mathrm{e}^{\mathrm{i}\varphi})\frac{1}{f(\varphi)}\mathrm{d}\varphi=0\quad\textit{for } j\in\left[-\frac{N}{2}+1,\dots,\left(m+\frac{1}{2}\right)N-1\right], \tag{3.10}$$

$$\int_{E_N}\mathrm{e}^{-\mathrm{i}j\varphi}\mathscr{U}_{(m-1)N}(\mathrm{e}^{\mathrm{i}\varphi})f(\varphi)\,\mathrm{d}\varphi=0\quad\textit{for } j\in\left[-\frac{N}{2}+1,\dots,\left(m-\frac{1}{2}\right)N-1\right], \tag{3.11}$$

i.e. $\{\mathscr{T}_{mN}\}_{m\in\mathbb{N}_0}$ resp. $\{\mathscr{U}_{(m-1)N}\}_{m\in\mathbb{N}}$ are the sequences of basic polynomials (with leading coefficients α^m resp. $\pm\alpha^{m-1}$) with respect to the weight functions f and $1/f$, respectively. Thereby we have set f, $1/f\equiv 0$ for $\varphi\notin E_N$.

Proof. First note that because of the simple zeros of r the function $1/f$ is integrable. To prove the theorem it suffices to show that with the setting

$$F(z):=\frac{\mathrm{i}z^{N/2}}{\sqrt{R_{2N}(z)}} \tag{3.12}$$

the function F satisfies

$$F(z)=\frac{1}{2\pi}\int_{E_N}\frac{\mathrm{e}^{\mathrm{i}\varphi}+z}{\mathrm{e}^{\mathrm{i}\varphi}-z}\frac{1}{f(\varphi)}\mathrm{d}\varphi,\quad |z|<1. \tag{3.13}$$

Then the orthogonality property (3.10) for $m\in\mathbb{N}$ follows from (3.13) and (1.8) by Theorem 2.1(a) and (3.9). From (3.12) we further see that F has a zero in $z=0$ of order $N/2$, thus in the power series expansion of F we have for the first $(\frac{1}{2}N+1)$ coefficients $c_0=\cdots=c_{N/2-1}=0$ and $c_{N/2}\neq 0$. This is (3.10) for $m=0$.

In [21, Lemma 3.1] we have shown that there holds

$$\lim_{\sigma\to 1-}\sqrt{R_{2N}(\sigma\mathrm{e}^{\mathrm{i}\varphi})}=\begin{cases}(-1)^j\mathrm{i}\mathrm{e}^{\mathrm{i}(N/2)\varphi}\sqrt{|r(\varphi)|}, & \varphi\in[\varphi_{2j-1},\varphi_{2j}],\ j=1,\dots,N,\\ (-1)^j\mathrm{e}^{\mathrm{i}(N/2)\varphi}\sqrt{|r(\varphi)|}, & \varphi\in[\varphi_{2j},\varphi_{2j+1}],\ j=0,\dots,N,\end{cases}$$

where again $\varphi_0 := a$ and $\varphi_{2N+1} := a + 2\pi$, thus we obtain

$$\lim_{\sigma\to 1-} \operatorname{Re} F(\sigma e^{i\varphi}) = \frac{1}{f(\varphi)} \in L_p(E_N) \quad \text{for all } p \in [0, 2[.$$

Since F is analytic on $|z| < 1$ (note that R_{2N} has all its zeros on $|z| = 1$ and N is even) with $F(0) = 0$ there can be derived from [19, Chap. I.D and V.B] that F has the representation (3.13). Together with (2.14) this proves the assertions concerning the $\mathscr{T}_{mN}$'s.

From (3.9) and Theorem 2.1(a) we see that $iz^{N/2}\mathscr{U}_{(m-1)N}$ is the polynomial of the second kind of $\mathscr{T}_{mN}$. Thus by Theorem 2.5 the $\mathscr{U}_{(m-1)N}$'s are orthogonal with respect to the function

$$G(z) := \frac{\sqrt{R_{2N}(z)} - i\Phi(z)}{iz^{N/2}},$$

where $\Phi \in \mathbb{P}_N^{\mathbb{C}}$ is defined as in (2.16) and satisfies $\operatorname{Re}(e^{-i(N/2)\varphi}\Phi(e^{i\varphi})) = 0$ for all $\varphi \in [0, 2\pi]$ since $\Phi^* = -\Phi$. Thus

$$\operatorname{Re} G(e^{i\varphi}) := \lim_{\sigma\to 1-} \operatorname{Re} G(\sigma e^{i\varphi}) = \begin{cases} f(\varphi) & \text{for } \varphi \in E_N, \\ 0 & \text{for } \varphi \notin E_N, \end{cases}$$

and (3.11) follows in the same way as (3.10). □

Remark. At the end let us mention that the results in this section also hold true (with minor modifications) if we choose $L \leqslant \min\{|t(\varphi)|: t'(\varphi) = 0\}$ in (3.1) and without the assumption N even. Then it can be shown (cf. [22]) that all the orthogonal polynomials with periodic reflection coefficients $\{a_n\}_{n\in\mathbb{N}_0}$, $|a_n| \neq 1$, can be represented resp. described by the polynomials from Lemma 3.1.

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